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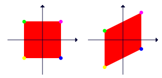
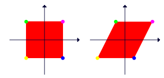
Lecture 8: Examples of linear transformations

While the space of linear transformations is large, there are few types of transformations which are typical. We look here at dilations, shears, rotations, reflections and projections.

Shear transformations

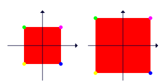
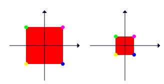
1

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ 

In general, shears are transformation in the plane with the property that there is a vector $\vec{w}$ such that $T(\vec{w}) = \vec{w}$ and $T(\vec{x}) - \vec{x}$ is a multiple of $\vec{w}$ for all $\vec{x}$. Shear transformations are invertible, and are important in general because they are examples which can not be diagonalized.

Scaling transformations

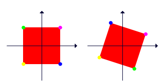
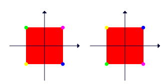
2

$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$  $A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ 

One can also look at transformations which scale x differently then y and where A is a diagonal matrix. Scaling transformations can also be written as $A = \lambda I_2$ where I_2 is the identity matrix. They are also called **dilations**.

Reflection

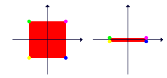
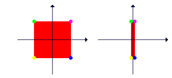
3

$A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$  $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ 

Any reflection at a line has the form of the matrix to the left. A reflection at a line containing a unit vector $\vec{u}$ is $T(\vec{x}) = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}$ with matrix $A = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$

Reflections have the property that they are their own inverse. If we combine a reflection with a dilation, we get a **reflection-dilation**.

Projection

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ 

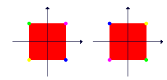
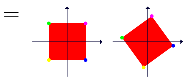
4

A projection onto a line containing unit vector $\vec{u}$ is $T(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u}$ with matrix $A = \begin{bmatrix} u_1u_1 & u_2u_1 \\ u_1u_2 & u_2u_2 \end{bmatrix}$.

Projections are also important in statistics. Projections are not invertible except if we project onto the entire space. Projections also have the property that $P^2 = P$. If we do it twice, it is the same transformation. If we combine a projection with a dilation, we get a **rotation dilation**.

Rotation

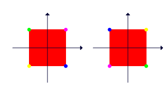
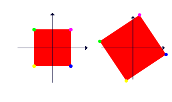
5

$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$  $A = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$ 

Any rotation has the form of the matrix to the right.
Rotations are examples of orthogonal transformations. If we combine a rotation with a dilation, we get a **rotation-dilation**.

Rotation-Dilation

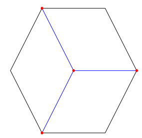
6

$A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$  $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ 

A rotation dilation is a composition of a rotation by angle $\arctan(y/x)$ and a dilation by a factor $\sqrt{x^2 + y^2}$.
If $z = x + iy$ and $w = a + ib$ and $T(x, y) = (X, Y)$, then $X + iY = zw$. So a rotation dilation is tied to the process of the multiplication with a complex number.

Rotations in space

7 Rotations in space are determined by an axis of rotation and an angle. A rotation by 120° around a line containing $(0,0,0)$ and $(1,1,1)$ belongs to $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ which permutes $\vec{e}_1 \to \vec{e}_2 \to \vec{e}_3$.

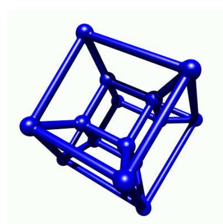


Reflection at xy-plane

8 To a reflection at the xy-plane belongs the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ as can be seen by looking at the images of $\vec{e}_i$. The picture to the right shows the linear algebra textbook reflected at two different mirrors.

Projection into space

9 To project a 4d-object into the three dimensional xyz-space, use for example the matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The picture shows the projection of the four dimensional cube (tesseract, hypercube) with 16 edges $(\pm 1, \pm 1, \pm 1, \pm 1)$. The tesseract is the theme of the horror movie "hypercube".



Homework due February 16, 2011

1 What transformation in space do you get if you reflect first at the xy-plane, then rotate around the z axes by 90 degrees (counterclockwise when watching in the direction of the z-axes), and finally reflect at the x axes?

2 a) One of the following matrices can be composed with a dilation to become an orthogonal projection onto a line. Which one?

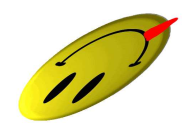
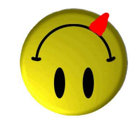
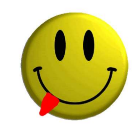
$$A = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

b) The **smiley face** visible to the right is transformed with various linear transformations represented by matrices $A - F$. Find out which matrix does which transformation:

$$\text{A=} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \text{B=} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \text{C=} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\text{D=} \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \quad \text{E=} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{F=} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}/2$$

A-F	image	A-F	image	A-F	image
					
					

3 This is homework 28 in Bretscher 2.2: Each of the linear transformations in parts (a) through (e) corresponds to one and only one of the matrices A) through J). Match them up.

a) Scaling b) Shear c) Rotation d) Orthogonal Projection e) Reflection

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} -0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix} \quad D = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix} \quad G = \begin{bmatrix} 0.6 & 0.6 \\ 0.8 & 0.8 \end{bmatrix} \quad H = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \quad I = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad J = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & -0.8 \end{bmatrix}$$