Mathematical Methods

Year 11 and 12

Unit A

A1: Functions and Graphs

- Functions
- Lines and linear relationships
- Quadratic relationships
- Polynomials and Powers
- Inverse proportion
- Graphs of relations

A2: Trigonometric Functions

- Cosine and sine rules
- Unit Circle
- Area of triangle
- Circular measure and radian measure
- Arc length and areas of sectors and segments in circles
- Exact values
- Graphs of $y = \sin x, y = \cos x$, and $y = \tan x$
- Amplitude
- Period
- Phase
- Angle sum and difference identities
- Trigonometric functions
- Modelling and problem solving

A3: Counting and Probability

Combinations

- Notation $\binom{n}{r}$ and the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- Expand $(x + y)^n$
- Binomial coefficients, $(ax + by)^n$
- Pascal's triangle and its properties
- Language of events and sets
- Sample spaces and events
- Union, Intersection, complement and mutually exclusive events
- Problems solving using Venn Diagrams

Fundamentals of probability

- likelihood of occurrence
- the probability scale: $0 \leq P(A) \leq 1$
- Addition Rule $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- Relative frequencies
- Conditional probability and independence
- $P(A|B)$, $P(A \cap B) = P(A|B)P(B)$
- Independence of an event A from an event B,
- Probability trees
- Independent events A and B, and symmetry of independence

Unit B

B1: Exponential Functions

Indices and the index laws:

- Index laws
- Fractional indices
- Scientific notation and significant figures
- Large and very small numbers
- Indicial equations and in equations
- The inverses of exponentials
- solve equations involving indices using logarithms

Exponential functions:

- Algebraic properties
- Sketch graphs, asymptotes, translations ($y = a^x + b$ and $y = a^{x+c}$) and dilations ($y = ba^x$)
- Modelling by exponential functions, practical problems
- solve equations and in equations

B2: Arithmetic and Geometric Sequence and Series

General sequences and number patterns

- Number Patterns
- Equations describing Fibonacci, triangular and perfect numbers
- Algebraic rules for number patterns
- Sigma notation for series

Arithmetic sequences:

- Recursive definition of an arithmetic sequence: $t_{n+1} = t_n + d$
- General form of AP $t_n = t_1 + (n-1)d$
- Applications in discrete linear growth or decay, including simple interest
- Sum of the first n terms of an arithmetic sequence

Geometric sequences:

- Recursive definition of a geometric sequence: $t_{n+1} = rt_n$
- General form of $t_n = r^{n-1}t_1$
- Limiting behaviour as $n \to \infty$ of the terms t_n and the common ratio r
- Limiting sum of a geometric sequence
- Sum of first *n* terms of GP sequence $S_n = t_1\frac{r^n-1}{r-1}$
- Applications of GP in geometric growth or decay, including compound interest and the determination of half-lives

B3: Introduction to Differential Calculus

Rates of change:

- Average rate of change $\frac{f(x+h)-f(x)}{h}$
- Leibniz notation δx and δy for changes or increments in the variables x and y
- Use $\frac{\delta y}{\delta x}$ for the difference quotient $\frac{f(x+h)-f(x)}{h}$ where $y = f(x)$
- Slope or gradient of a chord or secant of the graph of $y = f(x)$ as $\frac{f(x+h)-f(x)}{h}$ and $\frac{\delta y}{\delta x}$

The concept of the derivative:

- Continuity and discontinuity of functions, types
- Limits of functions from the left and from the right
- Concept of limit $\frac{f(x+h)-f(x)}{h}$ as $h \to 0$
- $f'(x)$ as $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
- $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ and the correspondence $\frac{dy}{dx} = f'(x)$ where $y = f(x)$
- Applications of rates of change --flow from different shaped vessels, and sketch graphs describing these rates
- Instantaneous rate of change
- Compare average and instantaneous rates of change
- Relationship between the graphs of f(x) and f'(x)

Computation of derivatives:

- Value of a derivative for simple power functions
- Variable rates of change of non-linear functions
- Establish the formula $\frac{d}{dx}(x^n) = nx^{n-1}$ for positive integers n by expanding $(x+h)^n$ or by factorising $(x+h)^n - x^n$
- Extend the formula $\frac{d}{dx}(x^n) = nx^{n-1}$ to apply for all rational n.

Properties of derivatives:

- Function is differentiable? Conditions for differentiability of a function
- Concept of the derivative as a function
- recognise and use linearity properties of the derivative
- Calculate derivatives of polynomials and other linear combinations of power functions.
- Stationary points and their nature from graphs of derivative functions and vice versa

Applications of derivatives:

- To find instantaneous rates of change
- To find the slope of a tangent and normal and the equation of the tangent and normal
- Interpret position-time graphs, with velocity as the slope of the tangent

- Sketch curves associated with polynomials;
- To find stationary points, and local and global maxima and minima;
- Examine behaviour as $x \to \infty$ and $x \to -\infty$
- Applications - optimisation problems arising in a variety of contexts involving polynomials on finite interval domains.
- Applications -- straight line motion graphs including those of position time, velocity and acceleration

Anti-derivatives:

- Anti-derivatives of polynomial functions and
- Applications to a variety of contexts including motion in a straight line.

Unit C

C1: Further Differentiation and Applications

Differentiation rules:

- Product and quotient rules
- Composition of functions and chain rule

Exponential functions:

- Gradient of $y = a^x$ at (0,1)
- Use the formulae $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}(a^x) = \log_e(a)\, a^x$
- Applications of exponential functions and their derivatives to solve practical problems.

Logarithmic functions:

- Natural logarithm $\ln x = \log_e x$
- Inverse relationship of the functions $y = e^x$ and $y = \ln x$
- Use the formulae $\frac{d}{dx}(\ln x) = \frac{1}{x}$ and $\frac{d}{dx}(\ln f(x)) = \frac{f\prime(x)}{f(x)}$
- Use the formulae $\int \frac{1}{x} dx = \ln|x| \ + c$ and $\int \frac{f\prime(x)}{f(x)} dx = \ln|f(x)| \ + c$
- Use logarithmic functions and their derivatives and integrals to solve practical problems.

Trigonometric functions:

- $\frac{d}{dx}(\sin x) = \cos x$, and $\frac{d}{dx}(\cos x) = -\sin x$
- use trigonometric functions and their derivatives to solve practical problems.

Further Differentiation:

- Applications of product, quotient and chain rule to differentiate functions such as xe^x, $\tan x$, $\frac{1}{x^n}$, $x \sin x$, $e^{-x} \sin x$, $\ln(\sin(x))$, $\ln(f(x))$, and $f(ax+b)$
- derivatives of the reciprocal trigonometric functions (sec, cosec, cot)

Second derivative and applications of differentiation:

- Concept of the second derivative as the rate of change of the first derivative function
- Acceleration as the second derivative of position with respect to time
- Concepts of concavity and points of inflection and their relationship with the second derivative
- Use of the second derivative test for finding local maxima and minima
- Graph of a function using first and second derivatives to locate stationary points and points of inflection
- Optimisation problems-- using first and second derivatives.

C2: Integrals

Anti-differentiation:

- Use the formula $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$ for $n \neq -1$
- Use the formula $\int e^x dx = e^x + c$
- Use the formula $\int \frac{f'(x)}{f(x)} dx = \ln(f(x)) + c$
- Integrals of the derivatives of trigonometric functions (including reciprocal functions)
- Use linearity of anti-differentiation
- Indefinite integrals of the form $\int f(ax+b)dx$
- Identify families of curves with the same derivative function
- Find $f(x)$, given $f'(x)$ and an initial condition $f(a) = b$
- Displacement given velocity in linear motion problems.

Definite integrals:

- Area problem, and use sums of the form $\sum_i f(x_i)\,\delta x_i$ to estimate the area under the curve $y = f(x)$
- Definite integral $\int_a^b f(x)dx$ as area under the curve $y = f(x)$ if $f(x) > 0$
- Definite integral $\int_a^b f(x)dx$ as a limit of sums of the form $\sum_i f(x_i)\,\delta x_i$
- Interpret $\int_a^b f(x)dx$ as a sum of signed areas
- Additivity and linearity of definite integrals

Fundamental theorem:

- Concept of the signed area function $F(x) = \int_a^x f(t)dt$
- Use the theorem: $F'(x) = \frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$, its proof geometrically
- Use the formula $\int_a^b f(x)dx = F(b) - F(a)$ and use it to calculate definite integrals

Applications of integration:

- Area bounded by a curve and either axis
- Total change by integrating instantaneous or marginal rate of change
- Area between curves in simple cases
- Determine positions given acceleration and initial values of position and velocity
- Volumes of solids of revolution formed by rotating simple regions around the x axis

C3: Discrete Random Variables

General discrete random variables:

- Concepts of a discrete random variable, probability function, and their use in modelling data
- Relative frequencies
- Uniform discrete random variables

- Non-uniform discrete random variables, for example Poisson and Hypergeometric distribution
- Mean or expected value of a discrete random variable as a measurement of centre, and evaluate it in simple cases
- Variance and standard deviation of a discrete random variable as a measures of spread, and evaluate them in simple cases
- Means and variances of linear combinations of random variables (e.g. (E(aX+b)=aE(X)+b, $\sigma^2_{aX+c} = a^2\sigma^2_X$, etc)
- Applications of discrete random variables and associated probabilities

Bernoulli distributions:

- use a Bernoulli random variable as a model for two-outcome situations
- identify contexts suitable for modelling by Bernoulli random variables
- recognise the mean p and variance $p(1-p)$ of the Bernoulli distribution with parameter p
- use Bernoulli random variables and associated probabilities to model data and solve practical problems.

Binomial distributions:

- Bernoulli trials and the concept of a binomial random variable
- Modelling by binomial random variables
- Binomial distribution
- Markov Chains
- Applications to model real-life data, drawing inferences from specific to general

Unit D

D1: The Logarithmic Functions

Logarithmic functions:

- Definition
- Algebraic properties of logarithms
- The inverse relationship between logarithms and exponentials
- Logarithmic scales –
    - decibels in acoustics,
    - the Richter Scale for earthquake magnitude,
    - octaves in music, and
    - pH in chemistry
- Graph of $y = \log_a x\ (a > 1)$ including asymptotes, and
- Translations $y = \log_a x + b$ and $y = \log_a(x + c)$
- solve equations involving indices using logarithms
- Equations involving logarithmic functions algebraically and graphically
- Applications of logarithmic functions to model and solve practical problems

Calculus of logarithmic functions:

- define the natural logarithm $\ln x = \log_e x$
- recognise and use the inverse relationship of the functions $y = e^x$ and $y = \ln x$
- establish and use the formula $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- establish and use the formula $\int \frac{1}{x} dx = \ln\ x\ + c$, for $x > 0$
- use logarithmic functions and their derivatives to solve practical problems

D2: Continuous Random Variables and the Normal Distribution

General continuous random variables:

- Relative frequencies and histograms
- Concepts of a probability density function, cumulative distribution function, and probabilities associated with a continuous random variable given by integrals
- Examine simple types of continuous random variables and use them in appropriate contexts
- Expected value, variance and standard deviation of a continuous random variable and evaluate them in simple cases
- Effects of linear changes of scale and origin on the mean and the standard deviation

Normal distributions:

- Modelling by normal random variables
- Graph of the probability density function of the normal distribution with mean μ and standard deviation σ and the use of the standard normal distribution

- Probabilities and quantiles associated with a given normal distribution using technology, and use these to solve practical problems.
- Calculate interval estimate of the mean (e.g. 95% confidence limits)
- Normal approximation to Binomial Distribution

D3: Interval Estimates and Proportions

Random sampling:

- Bias in samples, and procedures to ensure randomness
- Variability of random samples from various types of distributions, including uniform, normal and Bernoulli

Sample proportions:

- Concept of the sample proportion $\hat{p}$ as a random variable whose value varies between samples, and the formulas for the mean p and standard deviation $\sqrt{(p(1-p)/n}$ of the sample proportion $\hat{p}$
- Approximate normality of the distribution of $\hat{p}$ for large samples
- Simulate repeated random sampling

Confidence intervals for proportions:

- Interval estimate for a parameter associated with a random variable
- Quantile for the standard normal distribution
- Margin of error $E = z\sqrt{(\hat{p}(1-\hat{p})/n}$ and understand the trade-off between margin of error and level of confidence
- use simulation to illustrate variations in confidence intervals between samples and to show that most but not all confidence intervals contain p.

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